TRIGONOMETRY

All Functions on the Unit Circle

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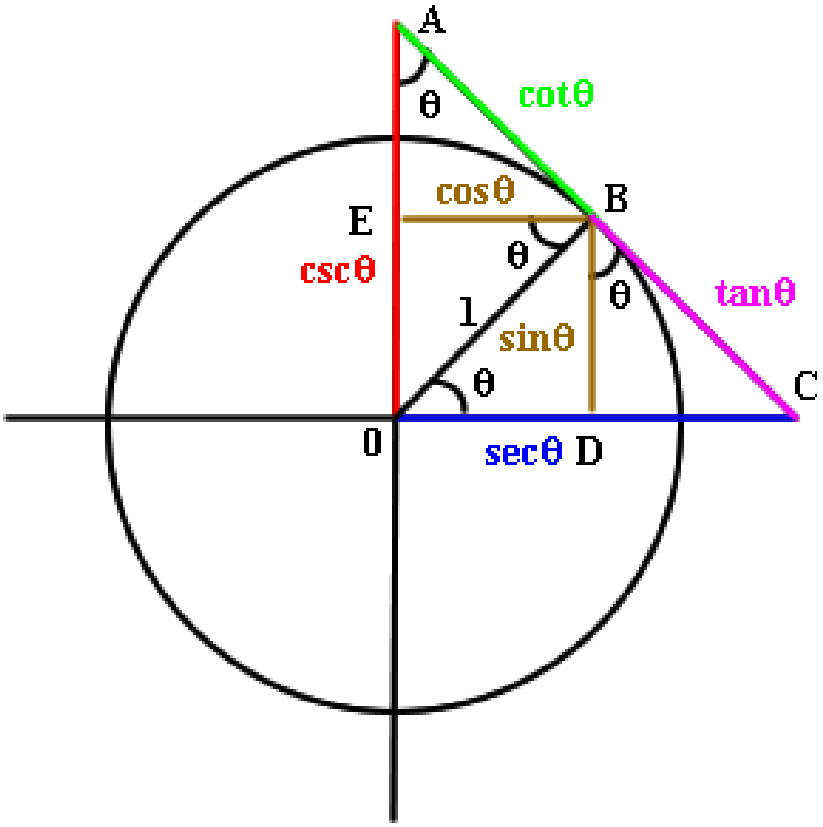


Until now, we have only seen the two basic functions (sin and cos) on the Unit Circle. However, we can also represent the other trig functions, tan, cot, sec and csc on the Unit Circle as shown above. But to prove that these lines equate to the trig functions of angle θ we have to understand the concept of Similar Triangles.

Similar Triangles

Similar triangles have the exact same shape but different sizes. In a similar triangle, the corresponding angles are equal and the ratio between corresponding sides are equal to a scale factor k, i.e., the sides are proportional. For two triangles to be similar —

- AA(Angle-Angle) Two angles of one triangle must be equal to two angles of another. Or,
- SAS(Side-Angle-Side) Two sides of one triangle must be equal to two sides of another triangle and the included angle must be equal. Or,

- SSS(Side-Side-Side) All corresponding sides of both triangles are equal or proportional

Here, we will mostly be using AA to prove two triangles are similar.

Let us take an example.

If △ABC ~ △DEF (similar) then —

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = k(scale\ factor)$$

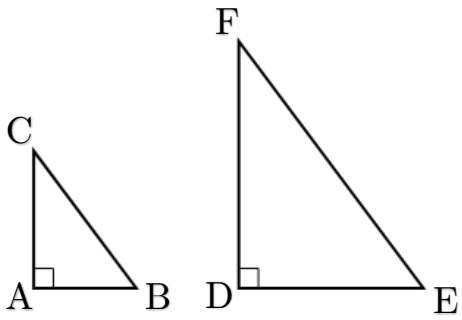
This shows that all corresponding sides are proportional.

But we can also say—

$$\frac{AB}{AC} = \frac{DE}{DF}$$

$$\frac{AC}{\boldsymbol{BC}} = \frac{DF}{EF}$$

$$\frac{AB}{BC} = \frac{DE}{EF}$$



This shows that the ratios between two corresponding sides are equal. As we know, the trigonometric functions are just ratios of the sides of a right triangle, so we will be using the relations between two corresponding sides of similar triangle to prove the functions on the Unit Circle.

Proof of sin and cos

We already know the proof of sin and cos but let us revise.

If Hyp = 1 then,

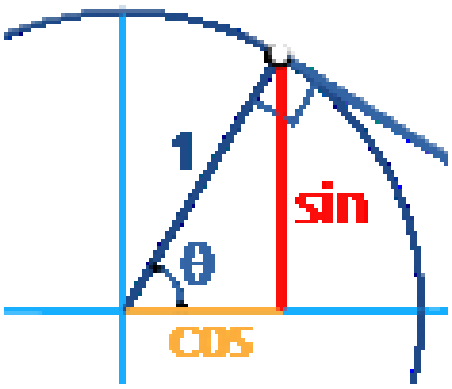
sinθ = Opp/Hyp

= Opp/1 = Opp

cosθ = Adj/Hyp

= Adj/1 = Adj

Thus, sinθ = Opp and cosθ = Adj



Proof of tan and cot

△ABD ~ △ACB (AA- θ and 90°)

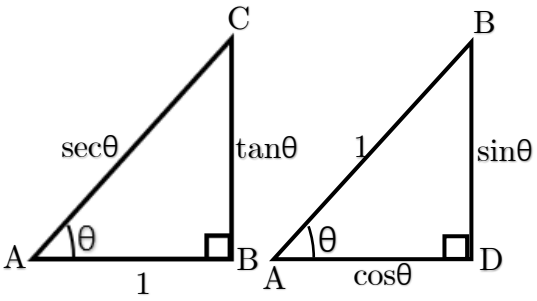
Thus,

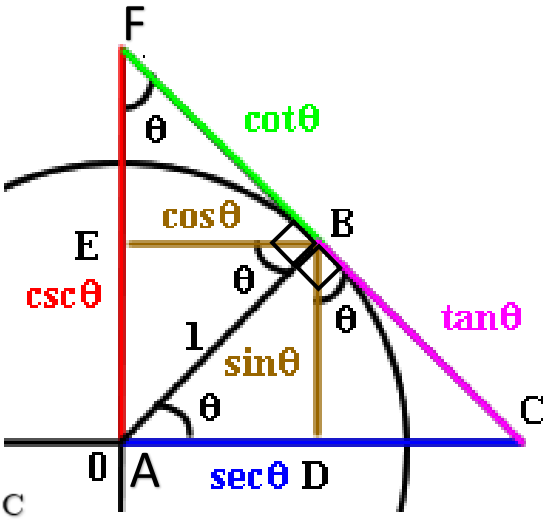
$$\frac{tan\theta}{1} = \frac{sin\theta}{cos\theta}$$

$$tan\theta = \frac{sin\theta}{cos\theta}$$

which is true.

And,



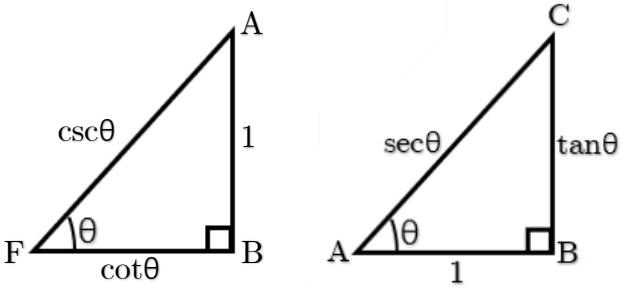


△FAB ~ △ACB (AA- θ and 90°)

$$\frac{cot\theta}{1} = \frac{1}{tan\theta}$$

$$cot\theta = \frac{1}{tan\theta}$$

which is true.



Proof of sec and csc

△ACB ~ △ABD (AA- θ and 90°)

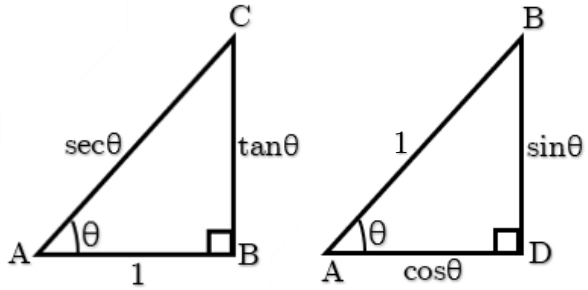
Thus,

$$\frac{sec\theta}{1} = \frac{1}{cos\theta}$$

$$sec\theta = \frac{1}{cos\theta}$$

which is true.

And,

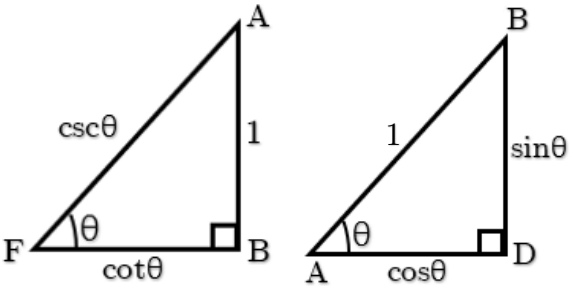


△FAB ~ △ABD (AA- θ and 90°)

$$\frac{csc\theta}{1} = \frac{1}{sin\theta}$$

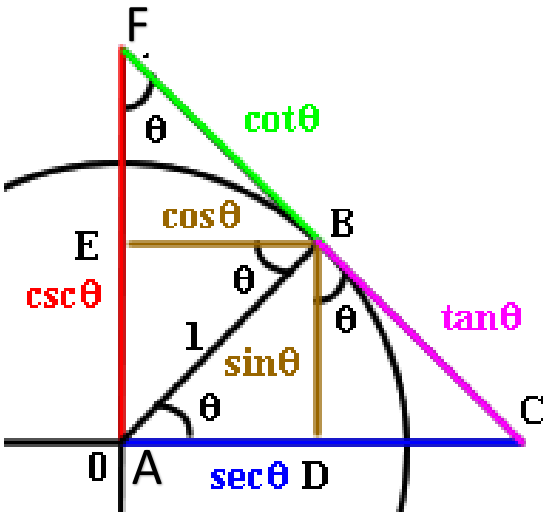
$$csc\theta = \frac{1}{sin\theta}$$

which is true.



Pythagorean theorems

Now that we have proved these trig functions, let us look at some interesting Pythagorean theorems that can be derived from them



In right triangle, △ABD —

$(cos\theta)^2 + (sin\theta)^2 = 1^2$

$cos^2\theta + sin^2\theta = 1$

This identity is perhaps one the most important in trigonometry; underlying the basic relation between sin and cos. But we can also derive —

In right triangle, △FAB —

$1^2 + (cot\theta)^2 = (csc\theta)^2$

$1 + cot^2\theta = csc^2\theta$

In right triangle, △ACB —

$(tan\theta)^2 + 1^2 = (sec\theta)^2$

$tan^2\theta + 1 = sec^2\theta$

These three are the most important identities. There are more identities, but they are not as well discussed as these three.

So, we conclude, with all the trig functions on the Unit Circle and their respective proofs. If you can visualise all the functions on the circle, finding the relationships and properties of the functions and their respective Pythagorean theorem becomes very easy, and useful when we will represent them on graphs, later.

Additional resources — https://youtu.be/Dsf6ADwJ66E ;visual animation of the 6 trig functions.